Stability exponents for smooth, recursive dynamical systems

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The linear stability of a dynamical system is governed by the variational equations. Making a linear change of variables in the variational equations and interpreting the transformation matrix to be a function of position in state space leads to a linear partial differential equation for the transformation matrix. The known solutions to the constant coefficient and periodic coefficient cases are shown to be natural consequences of this formulation. In the general case, for recursive dynamical systems a series of eigenvalue problems can be formulated, leading to stability exponents in the limit as time approaches infinity. A correlation method is offered which can reconstruct the missing multiples of 2π necessary to calculate the imaginary part. Numerical applications to the two-dimensional restricted three-body problem show that the imaginary parts of the new stability exponents are the winding numbers of the trajectory. In the three-dimensional restricted problem, the additional stability exponents match vertical oscillation frequencies.

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I. INTRODUCTION

A general dynamical system can be written as a set of vector differential equations

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}). \tag{1}$$

We have written the system (1) as an autonomous system. If it is not, defining $X_{N+1} \equiv t$ and appending $\dot{X}_{N+1} = 1$ to the above will make it autonomous. The system will be assumed to have this property from here onward. Once a trajectory $\mathbf{X}(t)$ is obtained, the question of stability naturally arises. If $\mathbf{x}$ represents the first order displacement from the known trajectory, then the *variational equations* specify the evolution of $\mathbf{x}$:

$$\dot{\mathbf{x}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right|_{\mathbf{X}(t)} \mathbf{x} = A(t)\mathbf{x}. \tag{2}$$

The properties of the solutions to (2) give the linear stability of the trajectory $\mathbf{X}(t)$ of the system (1). The solution to (2) can be written in terms of the fundamental matrix $\Phi(t, t_0)$ as

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0), \tag{3}$$

where the fundamental matrix itself obeys

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_0, t_0) = I. \tag{4}$$

Notice that the above is just a matrix form of (2), with special initial conditions.

The full solutions to the constant coefficient and periodic coefficient cases of (2) are known, and both of these cases admit of complex-valued stability exponents. In the general case, Lyapunov exponents determine linear stability properties, but have the unusual property of being purely real.

II. A LINEAR TRANSFORMATION

Consider making a change of variable from $\mathbf{x}$ to $\mathbf{y}$ in (2). If

$$\mathbf{x} = E(t)\mathbf{y}, \tag{5}$$

where $E(t)$ is a matrix, then substitution into the variational equations leads to

$$\dot{\mathbf{y}} = E^{-1}\left(AE - \dot{E}\right)\mathbf{y} = \Omega\mathbf{y}. \tag{6}$$

Both the transformation matrix E and the new system matrix Ω are as yet undetermined. For ease of solution of the new system $\dot{\mathbf{y}} = \Omega\mathbf{y}$, it would be very desirable if Ω could be a diagonal matrix. It would be even more desirable if Ω were constant, since then we could write the solution to (6) as

$$\mathbf{y}(t) = \exp\left[\Omega(t - t_0)\right]\mathbf{y}(t_0). \tag{7}$$

Using (7) and (5), and observing the initial condition on Φ from (4), this implies a factorization of the fundamental matrix in the form

$$\Phi(t, t_0) = E(t)\exp\left[\Omega(t - t_0)\right]E^{-1}(t_0). \tag{8}$$

Notice that since the E matrices to the right and left above are evaluated at different times, this is *not* the eigenvalue-eigenvector factorization studied by Wiesel [1]. To force all stability information into Ω, it is also desirable that $E(t)$ be strictly bounded. Rearranging (6) produces

$$\dot{E} = AE - E\Omega, \tag{9}$$

a relation that is familiar from Floquet theory. In Floquet's solution for time-periodic $A(t)$ systems, the $E(t)$

matrix is periodic (and therefore bounded), while Ω is (at worst) a constant Jordan form with Poincaré exponents as its diagonal entries.

However, suppose that we were to consider the matrix E not as a function of time, but as a function of position in the state space $\mathbf{X}$. This is trivially compatible with the constant coefficient case of (2), which arises at equilibrium points. The standard E matrix in this case is the matrix of eigenvectors of A and is associated with one point in state space $\mathbf{X}$. It is also compatible with the Floquet solution for periodic systems, where the periodic $E(t)$ matrix is associated with one particular point $\mathbf{X}(t)$ on the periodic trajectory. Since $E(\mathbf{X})$ is assumed nonsingular, this makes E a local coordinate basis at each point in the space. Note that not only is $\mathbf{f}$ given as a function of $\mathbf{X}$, but so is $A = \partial\mathbf{f}/\partial\mathbf{X}$ before its evaluation on any particular trajectory $\mathbf{X}(t)$. It is thus permissible to consider $A = A(\mathbf{X})$. The matrix Ω, on the other hand, is assumed to be constant, and a *property of the trajectory* underlying the linear system (2).

Assuming that the underlying dynamics are an autonomous system, the chain rule gives

$$\dot{E} = \frac{\partial E}{\partial \mathbf{X}}\dot{\mathbf{X}} = \frac{\partial E}{\partial \mathbf{X}}\mathbf{f}. \tag{10}$$

Then (9) becomes the first order linear matrix partial differential equation

$$\frac{\partial E}{\partial \mathbf{X}}\mathbf{f} = AE - E\Omega. \tag{11}$$

Furthermore, assuming Ω is diagonal, (11) can be written as

$$\frac{\partial \mathbf{e}}{\partial \mathbf{X}}\mathbf{f} = (A - \omega I)\,\mathbf{e}, \tag{12}$$

where $\mathbf{e}$ is one of the columns of E and ω is the corresponding element of the diagonal matrix Ω. (The assumption of diagonal Ω can be relaxed to permit a Jordan form if necessary.)

At this point, notice that we can recover the classical solution of the constant coefficient case from (12). Constant coefficient systems arise by linearizing about an equilibrium point, where $\mathbf{f} = 0$. Then the partial differential equation (12) reduces to

$$\mathbf{0} = (A - \omega I)\,\mathbf{e}, \tag{13}$$

which is just the algebraic eigenvalue problem. We can satisfy our goals for E and Ω by choosing the N $\mathbf{e}$'s to be the eigenvectors of A while the ω's will be the eigenvalues.

It is also quite simple to recover the Floquet solution. If the underlying trajectory is periodic in time, it must join itself in space. If $E = E(\mathbf{X})$ is really to be a function of position in state space, then E must have the same value at the end of one period as at the initial time. We are thus led quite naturally to demanding that $E(t) = E(\mathbf{X}(t))$ be a periodic matrix when evaluated along a periodic orbit. Then using (8) and the periodicity condition $E(\tau + t_0) = E(t_0)$ after one period τ, Floquet's eigenvalue problem

$$\Big[\Phi(\tau + t_0, t_0) - \exp(\omega\tau)I\Big]\mathbf{e}(t_0) = 0 \tag{14}$$

is a natural consequence. If the eigenvalues of the monodromy matrix are σ_i, we have the Poincaré exponents as

$$\omega_i = \frac{1}{\tau}\ln\sigma_i. \tag{15}$$

Then Eq. (9) can be used to construct the $E(t) = E(\mathbf{X}(t))$ matrix for one period, completing the Floquet solution. It is very interesting that *both* known solutions arise very naturally as special cases of (12) applied to equilibrium points and to periodic trajectories.

In the general case the solution is more complicated. However, one special solution to (12) is easily found. If the underlying dynamical system is autonomous, then $\mathbf{f}$ is a solution with $\omega = 0$. Direct substitution of $\mathbf{e} = \mathbf{f}$ into (12) gives

$$\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{X}}\mathbf{f} - \omega\mathbf{f}, \tag{16}$$

remembering that $A = \partial\mathbf{f}/\partial\mathbf{X}$. This is obviously an identity if $\omega = 0$.

III. PROXIMITY CONDITIONS

The solution to the constant coefficient case arises naturally from (12) at an equilibrium point. In the periodic coefficient case, the assumption that $E = E(\mathbf{X})$ forces the assumption that $E(t)$ is periodic on a periodic orbit and leads to Floquet's solution. In the general case, there are no such obvious special features. However, many dynamical systems are recursive, in the sense that the system returns repeatedly to a neighborhood of the initial point as $t \to \infty$. In particular, any trajectory confined to a bounded region of space without singularities must inevitably approach its initial conditions many times as $t \to \infty$. In these systems it is possible to extend Floquet's solution, using a "proximity condition" in place of the boundary condition $E(\tau + t_0) = E(t_0)$ in Floquet theory.

The fundamental difficulty with (12) is that it contains no information about the evolution of the $\mathbf{e}$ vectors in directions perpendicular to the $\mathbf{f}$ vector. However, when the trajectory returns to a neighborhood of its starting point, as shown in Fig. 1, this is no longer true. Let $t = t_c$ be a time when the trajectory returns to the neighborhood of the initial point and is closest to it. Then Eq. (8) can be recast as

$$\Phi(t_c, t_0)\mathbf{e}(t_0) = \mathbf{e}(t_c)\exp[\omega(t_c - t_0)]. \tag{17}$$

Now, since the trajectory has returned to the close proximity of the initial point, and assuming that $\mathbf{e} = \mathbf{e}(\mathbf{X})$ is smooth, we are justified in writing

$$\mathbf{e}(t_c) \approx \mathbf{e}(t_0) + \left.\frac{\partial \mathbf{e}}{\partial \mathbf{X}}\right|_{t_0}\delta\mathbf{X}(t_c), \tag{18}$$

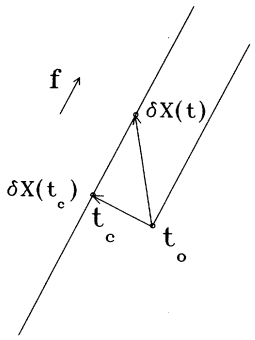

FIG. 1. Geometry of a return to a neighborhood of the initial point.

where $\delta\mathbf{X}$ is small. [Actually, this could well not be true. On a "strange attractor," it may well be that $E(\mathbf{X})$ is not locally smooth. We may speculate that the solutions to (12) may be "turbulent" in this case. On the other hand, orbits confined to a torus in a Hamiltonian system should admit smooth $E(\mathbf{X})$ solutions. We will confine our attention to such systems in this paper.] Combining these results produces

$$\{\Phi(t_c,t_0)-\exp[\omega(t_c-t_0)]I\}\mathbf{e}(t_0)$$

$$=\left.\frac{\partial\mathbf{e}}{\partial\mathbf{X}}\right|_{t_0}\delta\mathbf{X}(t_c)\exp[\omega(t_c-t_0)]. \quad (19)$$

Notice that as $\delta\mathbf{X}\to 0$, as it must eventually do as $t\to\infty$, the above reverts to a Floquet eigenvalue-eigenvector problem. Numerically, however, it is difficult to extend the calculation this far, particularly in systems where Φ grows unbounded. But in Hamiltonian systems in particular, $|\Phi|=1$ for all time.

Now, with a nonzero right-hand side, Eq. (19) makes no sense as an extension of the standard eigenvalue-eigenvector problem. If the right-hand side above is nonzero, then deterministic solutions for $\mathbf{e}(t_0)$ exist *for any value of ω which makes the matrix on the left-hand side nonsingular.* We are thus led to the approximation

$$\left.\frac{\partial\mathbf{e}}{\partial\mathbf{X}}\right|_{t_0}\delta\mathbf{X}(t_c)\exp[\omega(t_c-t_0)]\approx 0. \quad (20)$$

The only term in (19) which is "small" is the right-hand side. While rigorously true in the limit as $t\to\infty$, it is approximately true in any close approach to the initial state. Furthermore, the stability exponents ω calculated using this approximation will be insensitive to small changes in t_c, at least in the neighborhood of the initial state. However, we will be forced to use a finite neighborhood in numerical applications. If one of the ω has a positive real part, (20) will force the use of a neighborhood radius which *decreases exponentially in time.* This is almost certainly not practical, so we expect (20) will only be useful in systems with a full compliment of purely imaginary stability exponents.

With the condition (20) imposed, Eq. (19) becomes

$$\{\Phi(t_c,t_0)-\exp[\omega(t_c-t_0)]I\}\mathbf{e}(t_0)\approx 0, \quad (21)$$

which is Floquet's eigenvalue problem again. It is now only true in the limit as $t_c\to\infty$ and $\delta\mathbf{X}(t_c)\to 0$, however. The values of ω calculated from (21) for finite t_c with small but finite $\delta\mathbf{X}(t_c)$ will need some further work to find the true values of ω in the limit.

We are now in a position where some earlier assumptions can be justified. The desire to have Ω, and therefore the ω_i, constant and characteristic of the particular trajectory can be understood as follows. Consider Fig. 2, where close recursions occur over the intervals $(t_2,\ t_0)$ and $(t_3,\ t_1)$. Using the easily derived composition law for the fundamental matrix, the matrices over these two intervals can be written as

$$\Phi(t_2,\ t_0)=\Phi(t_2,\ t_1)\Phi(t_1,\ t_0), \quad (22)$$
$$\Phi(t_3,\ t_1)=\Phi(t_3,\ t_2)\Phi(t_2,\ t_1).$$

Now, for trajectories confined to a smooth torus, the fact that the trajectories are within distance ϵ at t_2 implies that they remain close over the two intervals $(t_0,\ t_1)$ and $(t_2,\ t_3)$. (Notice that this assumption rules out applying this argument to strange attractors, where adjacent trajectories can depart exponentially from each other.) Since these two arcs of the trajectory lie very close to each other, we can be assured that

$$\Phi(t_1,\ t_0)\approx\Phi(t_3,\ t_2). \quad (23)$$

Then, consider

$$\Phi(t_1,\ t_0)^{-1}\Phi(t_3,\ t_1)\Phi(t_1,\ t_0)\approx\Phi(t_2,\ t_1)\Phi(t_1,\ t_0)$$
$$=\Phi(t_2,\ t_0). \quad (24)$$

This shows that the fundamental matrices $\Phi(t_2,\ t_0)$ and $\Phi(t_3,\ t_1)$ are approximately similar. In the limit as $t\to\infty$, this approximation will become better and better. Now, similar matrices have the same eigenvalues, so the ω values calculated from these different arcs will be, in the limit, the same. The Ω matrix should be constant and a property of the underlying trajectory.

A similar argument can be made that the ω's calculated over longer and longer intervals should approach a limit. In Fig. 3 a second close approach occurs after ap-

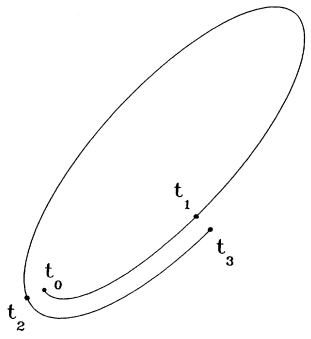

FIG. 2. Invariance of the stability exponents to initial starting time.

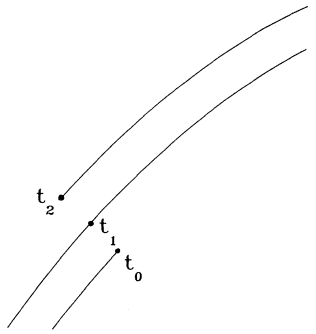

FIG. 3. Calculation of the fundamental matrix at a second close approach in which the period approximately doubles.

proximately double the initial time interval. Again, the two arcs of the trajectory lie close together, so

$$\Phi(t_2,\ t_0) = \Phi(t_2,\ t_1)\Phi(t_1,\ t_0) \approx \Phi(t_1,\ t_0)\Phi(t_1,\ t_0). \tag{25}$$

Writing $\Phi(t_1,\ t_0) = E\exp[\Omega(t_1 - t_0)]E^{-1}$, we have

$$\Phi(t_2,\ t_0) \approx E\exp[\Omega(t_1 - t_0)]\exp[\Omega(t_1 - t_0)]E^{-1} = E\exp[2\Omega(t_1 - t_0)]E^{-1}. \tag{26}$$

In words, this says that the stability exponents calculated over the (approximately) twice as long interval should be approximately the same as over the shorter interval. The above also shows that the **e** vectors should also be nearly identical. This argument easily extends to other near multiples of the "period." Once again, it is the fact that the trajectories will lie in close proximity over their length that makes this conclusion possible. Also, as before, on chaotic structures this may not be true.

IV. NUMERICAL METHODS

Let the initial conditions for a trajectory be $\mathbf{X}(t_0)$, and let the state at time t be $\mathbf{X}(t)$. The distance between the initial and current state is

$$d(t) = \|\mathbf{X}(t) - \mathbf{X}(t_0)\|, \tag{27}$$

where $\|\ \|$ is a norm. If the trajectory is bounded in space, then the function $d(t)$ must experience a series of maxima and minima. Define a sequence of *almost periods* τ_i as time intervals $\tau_i = t - t_0$, where $d(t)$ experiences a minimum value which is *less than its previous smallest minimum value.* We denote these minimum distances $d(\tau_i) = \epsilon_i$. On this sequence, we have

$$\lim_{i\to\infty} \epsilon_i = 0, \quad \lim_{i\to\infty} \tau_i = \infty. \tag{28}$$

The orbit comes closer and closer to actually closing on itself. This is true mathematically, but in numerical calculation error can accumulate, perhaps causing the distance $d(t)$ to become bounded from below. An alternate approach, and one that we will follow, is to pick a suitably small ϵ, and then process all times τ_i at which $\dot{d}(t) = 0$ and $d(t) \le \epsilon$.

We are thus led to a sequence of Floquet problems for the sequence of almost periods. If the sequence of $\omega_j(\tau_i)$ calculated for the almost period sequence τ_i converge to limits, then we are justified in calling the $\omega_j(\infty)$ the stability exponents of the trajectory. In fact, they will be Lyapunov exponents. Unfortunately, the imaginary part of the stability exponent calculated from (15) and (21) is uncertain to within an arbitrary multiple of 2π. If we do not try to resolve this difficulty, since the imaginary part of the natural logarithm obeys $-\pi \le \mathrm{Im}\ln x \le \pi$, we have

$$|\mathrm{Im}\ \omega_j| \le \frac{\pi}{\tau_i} \to 0 \tag{29}$$

as $\tau_i \to \infty$. But this hardly constitutes a proof that the imaginary parts are zero. In fact, applying the above on a periodic orbit at successive multiples of the true period would lead to a zero imaginary part, even if the actual Poincaré exponent does have an imaginary part. In order to calculate an imaginary part over long time intervals, the correct Reimann sheet must somehow be used in the complex logarithmic function.

Assume that we have found N almost periods and have calculated the eigenvalues $\sigma_j(\tau_i)$ of the fundamental matrix Φ at these times. Considering (15) and (21), their logarithms should be approximate values of

$$\ln\sigma_j(\tau_i) \approx \omega_j\tau_i. \tag{30}$$

(The approximation arises because we are, of necessity, dealing with finite times.) In practice, the values of ω_i calculated at different almost periods will be discordant, since unknown multiples of 2π are missing on the left-hand side of the above. To make progress in reconstructing the missing phase information, consider calculating $\mathrm{Mod}(\omega\tau_i)$, where the function Mod reduces the range of the imaginary part of its argument to $(-\pi,\ \pi)$. This is, of course, just the range produced by the complex logarithmic function. Then $\ln\sigma_j(\tau_i) - \mathrm{Mod}(\omega\tau_i)$ should be small when ω (here treated as a free variable) is close to the true value of ω_i.

Now consider $\ln\sigma_j(\tau_i) - \mathrm{Mod}(\omega\tau_i)$ as a residual. Among the different eigenvalues σ_j, there is one eigenvalue which minimizes this residual. Then calculate the root-mean-square value $\Gamma(\omega)$ of the residual as a function of ω using the minimizing eigenvalue σ_j over many almost periods:

$$\Gamma(\omega) = \left\{\frac{1}{N}\sum_{i=1}^{N}\{\mathrm{Min}_j\,[\ln\sigma_j(\tau_i) - \mathrm{Mod}(\omega\tau_i)]\}^2\right\}^{1/2}, \tag{31}$$

where Min_j chooses the minimum value of the residual among the different eigenvalues σ_j of the fundamental matrix. Considered as a function of ω, Γ should be of order unity when ω is far from the true value of ω_i and should be small when $\omega \approx \omega_i$. Deep minima in $\Gamma(\omega)$ should mark the true values of ω.

V. NUMERICAL EXPERIMENTS

As a first application, we consider the restricted problem of three bodies discussed by Szebehely [2]. In the usual dimensionless units, the system Hamiltonian takes the form

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 q_2 - p_2 q_1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \qquad (32)$$

where μ is the mass parameter. The two radii from the third body to the primaries are

$$r_1^2 = (q_1 - \mu)^2 + q_2^2, \qquad (33)$$
$$r_2^2 = (q_1 + 1 - \mu)^2 + q_2^2.$$

As is well known, this problem possesses periodic orbits, invariant torii, and chaotic regions.

As a Hamiltonian system with a four-dimensional phase space and a constant of the motion, it is also possible to construct surfaces of section. Figure 4 shows a portion of such a surface constructed using the conventions of Jefferys [3] for $\mu = 1/3$ and $H = -1.688\,888\,888\,88$. The section criterion is

$$q_1 \dot{q}_1 + q_2 \dot{q}_2 = 0. \qquad (34)$$

Knowing values of q_1 and q_2, the value of the Hamiltonian and the section condition supply the values of both momenta. The central periodic orbit is surrounded by invariant tori, which terminate in a chaotic region. Embedded within the main structure is another periodic orbit and its associated tori, with a period about three times that of the main orbit. Finally, a small island at about seven times the fundamental period appears just outside the point where the main island transitions to chaos. The figure shows only one of four intersections of this structure with the plane of section.

The periodic orbits (both stable and unstable) have been constructed by solving the numerical boundary value problem. Then Floquet theory can be used to calculate Poincaré exponents. Since this is a Hamiltonian system with an integral of the motion, one pair of Poincaré exponents is zero. The other pair is purely imaginary on both stable periodic orbits. In this type of system, the winding number (or rotation number) can also be calculated, as done by Guckenheimer and Holmes [4]. This is just the average rotation rate of points as they successively intersect the surface of section, as seen from the central periodic orbit. Of course, in the 3:1 resonant island, the central periodic orbit for that torus must be used. The winding number agrees with the Poincaré exponent as either stable periodic orbit is approached.

We have numerically integrated many orbits in this torus, calculating the eigenvalues of Φ at close approaches to the original conditions. The integrations were usually to a final time of $t = 1000$, and any approaches within $\epsilon = 0.03$ in the four-dimensional phase space were used. Plotting the components of the $\mathbf{e}(t_0)$ vectors as a function of $\delta\mathbf{X}(t_c)$, it appears that the $\mathbf{e}$ vectors are indeed smooth functions of position in the neighborhood of the initial state. As the fundamental matrix of a Hamiltonian system is symplectic [5], the logarithms of its eigenvalues $\ln\sigma_i$ must occur as positive-negative imaginary pairs. In calculating the root-mean-square residual $\Gamma(\omega)$, real root pairs were ignored. Figure 5 shows one plot of $\Gamma(\omega)$, with a very obvious high correlation at about $\omega = 0.6636$. This is the winding number for this trajectory. Other less strong correlations occur near the frequency of the orbit plus or minus the winding number. The correlation does not have to be perfect, since in (20) we have approximated small but finite $\delta\mathbf{X}(t_c)$ as zero. This introduces error that should vanish as $t \to \infty$.

Figure 6 shows the winding number (as a solid curve) and the imaginary stability exponent ω (circles) as a function of initial condition q_1 (with $q_2 = 0$). (Of course, values of ω occur as positive-negative imaginary pairs, since they are constructed from the eigenvalues of a symplectic Hamiltonian Φ matrix. Only the positive value is shown.) The figure should be compared with Fig. 4 sliced along the q_1 axis. The 3:1 resonant island is easily recognized as the separated region in the left center

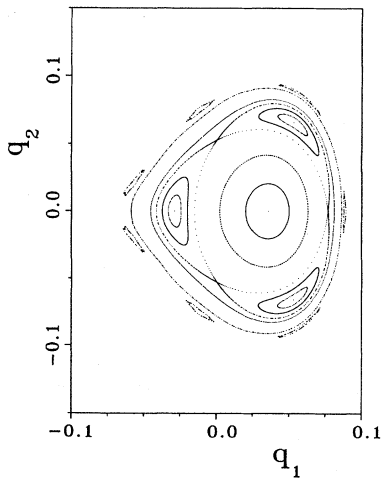

FIG. 4. Surface of section for the restricted problem.

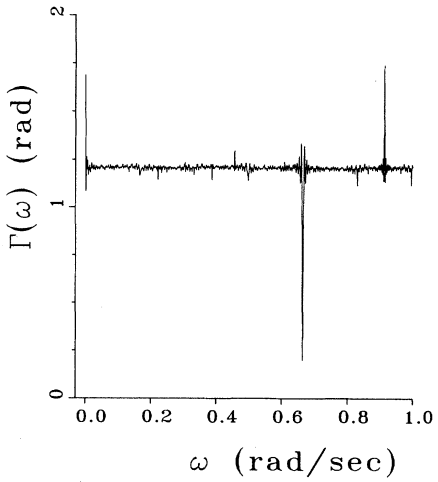

FIG. 5. Root-mean-square residual Γ as a function of ω for an orbit with initial conditions $q_1 = 0.04$, $q_2 = 0$. The large dip occurs at the winding number for this trajectory.

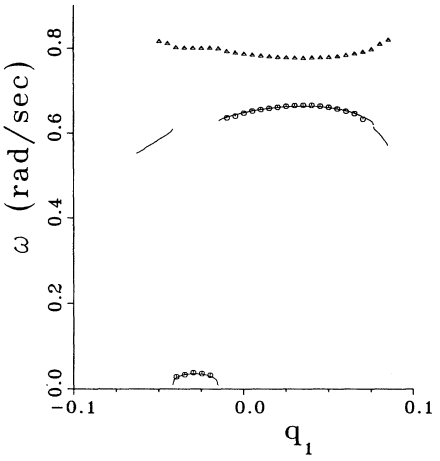

FIG. 6. Winding number (solid line) and stability exponents ω as a function of initial q_1 value, with an initial $q_2 = 0$. Planar ω values are shown as circles, vertical mode ω values as triangles.

of the plot. Throughout the central torus and the 3:1 island, the winding number and the stability exponent ω agree quite well. In some regions of this figure there are no values shown for ω. In these areas there were so few values of $\ln \sigma_i$ appearing among the calculated close approaches to the initial conditions that the correspondence technique failed to produce a distinct ω value. Everywhere else, the correspondence between the winding number and the stability exponent ω is quite good.

The correspondence of winding numbers with the stability exponents ω defined in this paper is quite encouraging, but the real question is whether our technique can be extended to higher-dimensional phase spaces. Winding numbers are a geometrical concept and are limited to systems in which a Poincaré surface of section can be constructed. For Hamiltonian systems, this requires a system with a four-dimensional phase space and one integral of the motion, exactly like the restricted three body problem. In searching for a higher-dimensional test case, it would also be helpful if a system could be found in which the frequencies are *already known or can be verified.*

We need look no further than the three-dimensional restricted three-body problem to meet these needs. The Hamiltonian is given by

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + p_1 q_2 - p_2 q_1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{35}$$

with radius vectors

$$r_1^2 = (q_1 - \mu)^2 + q_2^2 + q_3^2, \tag{36}$$
$$r_2^2 = (q_1 + 1 - \mu)^2 + q_2^2 + q_3^2.$$

We now have a six-dimensional phase space and will now have two additional Poincaré exponents at the periodic orbits, but there is still only one exact integral of the motion. Winding numbers cannot be constructed in this case. Furthermore, for infinitesimal q_3 values the z dynamics decouple from the motion in the q_1-q_2 plane. New stability exponents can then be checked by Fourier transforms of the q_3 components of orbits nearly in the q_1-q_2 plane.

Figure 6 also shows the ω values (as triangles) for the vertical motion of the orbits in this structure. The value of ω at the central periodic orbit matches that of the Poincaré exponent, while at the stable 3:1 orbit a multiple of 2π must be added to the logarithm in (15) to match the adjacent torus and the value of ω. Of course, this is quite permissible. The calculated ω points then delineate a continuous function across this structure. Fourier transforms of the q_3 component of trajectories very near the planar orbits shown here confirm that the calculated ω values are indeed the vertical oscillation frequencies.

VI. DISCUSSION AND CONCLUSIONS

The introduction of a linear change of variables to the equations of variation leads naturally to a partial differential equation for the transformation matrix when that matrix is interpreted to be a function of position in space. This interpretation leads to a very natural recovery of the constant coefficient and periodic coefficient cases as special cases of the general linear system. In the general case, stability exponents can be defined in recursive dynamical systems as the "Poincaré " exponents of the fundamental matrix at close approaches to the initial conditions. It has been argued that these exponents should be invariant to changes in starting point along the trajectory.

In numerical experiments, a correlation technique has been offered that is very successful in reconstructing the correct multiples of 2π necessary to correctly construct the imaginary part of an "extended" Lyapunov exponent. In applications to the planar restricted problem of three bodies, the imaginary part of the stability exponent ω appears to be identical to the winding number of the trajectory. Extending the calculations to three dimensions shows that the technique offered in this paper is able to calculate imaginary stability exponents even when a winding number cannot be defined.

One of the advantages of the method is that it extends to higher orders, where geometric methods such as the surface of section are not applicable. Indeed, a successful application to a system of order 18 has already been reported by Wiesel [6], and further work is in progress.

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